

A new C -integrable limit of second harmonic generation equations

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February 8, 2008

Abstract

A new C -integrable limit of the second harmonic generation equations is found. The corresponding general solution is given in an explicit form. Connection of this problem with the modified Liouville equation is discussed.

Second harmonic generation (SHG) is a well-known method of experimental investigations in nonlinear optics. Under idealized condition of only one space dimension this process is described by the equations (see, e.g., [1])

$$q_{1\chi} = -2q_2q_1^*, \quad q_{2\tau} = q_1^2, \quad (1)$$

where differentiation with respect to characteristic variables χ and τ is indicated by subscripts, the asterisk denotes complex conjugation, and q_1 and q_2 are slowly varying complex electric field amplitudes of two electromagnetic waves with carrier frequencies ω_1 and $\omega_2 = 2\omega_1$, respectively. The SHG equations (1) are shown to be “ S -integrable” [2] what permitted one to find some particular solutions describing transformation of waves. However, application of the inverse scattering transform method to the Cauchy problem meets some peculiar difficulties and is not fully developed yet. Therefore, other approaches to solving the SHG equations are of considerable interest.

As it was noticed long ago [3], in the case of pure amplitude modulation, when q_1 and q_2 are real variables, the SHG equations can be reduced to the C -integrable Liouville equation, so that the solution can be expressed explicitly in terms of two arbitrary functions. Some particular examples of such solutions were studied in [4], and the Liouville solution was applied to the so-called restricted Cauchy problem typical for experimental situation in [5].

Here we want to note that the SHG equations (1) can be approximated under certain condition by the system which is C -integrable and its solution is given in an explicit form in terms of four arbitrary functions. This limit is in a sense opposite to the discussed previously pure modulation case. (Note, however, that a pure modulation case is an exact reduction

of the SHG equations, whereas here we consider some approximation to these equations.) The proposed approximation is based on the following observation. Elimination of q_2 from Eqs. (1) leads to the equation

$$q_{1,\chi\tau}q_1^* - q_{1,\chi}q_{1,\tau}^* = -2(q_1q_1^*)^2, \quad (2)$$

where the left hand side is quadratic and the right hand side is of fourth degree in the amplitude q_1 . Hence, one may suggest that in the limit of small enough $|q_1|$ we can neglect the right hand side, if derivatives of q_1 in the left hand side are not too small. To formulate the corresponding criterion, it is convenient to pass to real variables introduced in [6]. We represent q_1 in the form

$$q_1 = \left(\sqrt{Q}/2\right) \exp(i\phi/2), \quad Q > 0, \quad (3)$$

so that q_2 is given by

$$q_2 = -\frac{1}{4} \left[(\ln Q)_\chi + i\phi_\chi \right] \exp(i\phi), \quad (4)$$

and Eq. (2) reduces to the system

$$(\ln Q)_{\chi\tau} - \phi_\chi\phi_\tau = -Q, \quad (Q\phi_\tau)_\chi = 0. \quad (5)$$

Introducing the variables

$$s = 1/Q, \quad u = \phi_\tau, \quad v = (\ln s)_\chi, \quad w = \phi_\chi \quad (6)$$

we rewrite the system (5) in the form [6]

$$(s)_\chi = sv, \quad u_\chi = w_\tau = vu, \quad v_\tau = 1/s - uw. \quad (7)$$

If we put here $u = 0$, we obtain at once the Liouville equation $(\ln s)_{\chi\tau} = 1/s$ which application to SHG was considered in [3, 5]. On the other hand, if $1/s \ll |uw|$, or

$$Q \ll |\phi_\tau\phi_\chi|, \quad (8)$$

we arrive at the system

$$v_\tau = -uw, \quad w_\tau = uv, \quad u_\chi = uv, \quad (9)$$

whereas s can be found from the equation $(\ln(s/u))_\chi = 0$ which follows from Eqs. (7), i.e.

$$s = K(\tau)u, \quad (10)$$

where $K(\tau)$ is an arbitrary function.

To find the general solution of the system (9), we eliminate v and w ,

$$v = \frac{u_\chi}{u}, \quad w = -\frac{1}{u} \left(\frac{u_\chi}{u} \right)_\tau = -\frac{u_{\chi\tau}}{u^2} + \frac{u_\chi u_\tau}{u^3}, \quad (11)$$

so that w_τ can be presented in the form

$$w_\tau = -\frac{1}{u} \left(\frac{u_{\tau\tau} - \frac{3}{2}(u_\tau^2/u)}{u} \right)_\chi.$$

According to Eq. (9), this must be equal to u_χ , hence we obtain

$$\left(\frac{u_{\tau\tau} - \frac{3}{2}(u_\tau^2/u)}{u} \right)_\chi = -\frac{1}{2} (u^2)_\chi,$$

or

$$u_{\tau\tau} - \frac{3}{2} \frac{u_\tau^2}{u} = -\frac{1}{2} u^3 + f(\tau)u, \quad (12)$$

where $f(\tau)$ is an arbitrary function. Generally speaking, this equation cannot be solved for a given $f(\tau)$ in an explicit form. However, in our case $f(\tau)$ is an arbitrary function and it can be replaced by another arbitrary function $F(\tau)$ related with $f(\tau)$ in the following way. We make a substitution

$$u = F'(\tau)U(\xi), \quad \xi = F(\tau), \quad (13)$$

so that Eq. (12) reduces to

$$U_{\xi\xi} - \frac{3}{2}(U_\xi^2/U) + \frac{1}{2}U^3 = \frac{1}{(F'(\tau))^2} \left[f(\tau) + \frac{3}{2} \left(\frac{F''(\tau)}{f'(\tau)} \right)^2 - \frac{F'''(\tau)}{F'(\tau)} \right] U.$$

Thus, if $f(\tau)$ is expressed in terms of $F(\tau)$ by the equation

$$f(\tau) = \frac{F'''(\tau)}{F'(\tau)} - \frac{3}{2} \left(\frac{F''(\tau)}{f'(\tau)} \right)^2 \equiv \{F, \tau\}, \quad (14)$$

where the curly bracket denotes the Schwarzian derivative [7], then $U(\xi)$ is determined by the equation

$$U_{\xi\xi} - \frac{3}{2}(U_\xi^2/U) + \frac{1}{2}U^3 = 0. \quad (15)$$

This equation is solved by elementary methods to give

$$U(\xi) = [1/G + \frac{1}{4}(\xi - H)^2]^{-1},$$

where G and H are integration constants (arbitrary functions of χ). Making use of Eqs. (13) and (11), we arrive at the general solution of the system (9):

$$\begin{aligned} u(\chi, \tau) &= \frac{4F'G}{4 + G^2(F - H)^2}, \\ v(\chi, \tau) &= \frac{-GG'(F - H)^2 + 2G^2H'(F - H) + 4G'/G}{4 + G^2(F - H)^2}, \\ w(\chi, \tau) &= \frac{\frac{1}{2}G^3H'(F - H)^2 + 4G'(F - H) - 2GH'}{4 + G^2(F - H)^2}, \end{aligned} \quad (16)$$

where $F(\tau)$, $G(\chi)$, $H(\chi)$ are arbitrary functions. Together with Eq. (10), these formulas give full description of the SHG problem in the limit (8).

From mathematical point of view, the fact of C -integrability of the system (9) is not trivial and finds its explanation in possibility to reduce this system to the so-called modified Liouville equation (see Appendix).

One may hope that this solution will permit one to give analytic description of SHG processes for concrete examples in a way similar to that used in [5].

We thank A.V. Zhiber for indication of Ref. [8] and H. Steudel for useful remarks on the manuscript. AMK is grateful to DFG (grant 436 RUS 113/89/2 (R,S)) and INTAS (grant 96-0339) for partial support. MVP is grateful to RFBR (grants 00-01-00210 and 00-01-00366).

Appendix

Let us show that the system (9) can be reduced to the modified Liouville equation first introduced by E. Vessiot [8] (see also the paper [9]).

We notice that the last two equations (9) yield a first integral

$$v^2 + w^2 = (\psi'(\chi))^2, \quad (\text{A.1})$$

where $\psi(\chi) > 0$ is an arbitrary function. Then transition to polar coordinates

$$v = \psi'(\chi) \cos \theta, \quad w = \psi'(\chi) \sin \theta, \quad (\text{A.2})$$

transforms the last two equations (9) to

$$u = \theta_\tau, \quad (\text{A.3})$$

and the first equation (9) takes the form $\theta_{\tau\chi} = \psi'(\chi) \cos \theta \cdot \theta_\tau$ or

$$\theta_{\tau\zeta} = \cos \theta \cdot \theta_\tau, \quad (\text{A.4})$$

where $\zeta = \psi(\chi)$. (This equation can be integrated once and reduced to a linear second order equation; see Eq. (A.20) below.) Introducing a new variable

$$\phi = \ln u = \ln \theta_\tau, \quad (\text{A.5})$$

we obtain

$$\phi_\zeta = \theta_{\tau\zeta}/\theta_\tau = \cos \theta, \quad \phi_{\zeta\tau} = -\sin \theta \cdot \theta_\tau,$$

and finally

$$\phi_{\tau\zeta} = -e^\phi \sqrt{1 - \phi_\zeta^2}, \quad (\text{A.6})$$

which is just the modified Liouville equation [8]. (Amplitude-modulated reduction of degenerate two-photon propagation equations discussed in [10] can be also reduced to this equation.)

This observation permits us to write down the solution of Eq. (A.6). Indeed, the general solution (16) gives

$$(\psi'(\chi))^2 = v^2 + w^2 = (G'/G)^2 + \frac{1}{4}(GH')^2.$$

Hence, if we choose $\psi'(\chi) = 1$, so that $\zeta = \chi$ and $H(\chi)$ is connected with $G(\chi)$ by the equation

$$(H')^2 = \frac{4}{G^4}[G^2 - (G')^2], \quad (\text{A.7})$$

then the solution of Eq. (A.6) is given by

$$\phi = \ln u = \ln \frac{4F'G}{4 + G^2(F - H)^2}. \quad (\text{A.8})$$

Note also that another system (arisen first in a physical problem about amplification of two counter propagating light beams by transverse flow of an amplifying medium)

$$a_\tau = ac, \quad b_\tau = -bc, \quad c_\chi = -(a + b)c,$$

whose general solution was found long ago [11], can be reduced by substitutions

$$u = c, \quad w = b - a, \quad v = -(a + b),$$

to the system

$$v_\tau = uw, \quad w_\tau = uv, \quad u_\chi = uv, \quad (\text{A.9})$$

which differs from Eqs. (9) only by the sign in the right hand side of the first equation, and is connected with another forms of the modified Liouville equation

$$\phi_{\tau\zeta} = e^\phi \sqrt{\phi_\zeta^2 \pm 1}, \quad (\text{A.10})$$

depending on the choice of the sign in the first integral

$$v^2 - w^2 = \mp (\psi'(\chi))^2. \quad (\text{A.11})$$

General solution of Eqs. (A.9) is given by the formulas

$$\begin{aligned} u(\chi, \tau) &= \frac{4F'G}{4 - G^2(F - H)^2}, \\ v(\chi, \tau) &= \frac{GG'(F - H)^2 - 2G^2H'(F - H) + 4G'/G}{4 - G^2(F - H)^2}, \\ w(\chi, \tau) &= \frac{-\frac{1}{2}G^3H'(F - H)^2 + 4G'(F - H) - 2GH'}{4 - G^2(F - H)^2}. \end{aligned} \quad (\text{A.12})$$

Then we have

$$v^2 - w^2 = G'^2/G^2 - \frac{1}{4}G^2H'^2,$$

and

$$\phi = \ln \frac{4F'G}{4 - G^2(F - H)^2} \quad (\text{A.13})$$

is the solution Eqs. (A.10) provided H satisfies the equation

$$H'^2 = 4[G'^2 \pm G^2]/G^4 \quad (\text{A.14})$$

with a proper choice of the sign.

As was noted in [8], solutions of the modified Liouville equations (A.6) are connected with solutions of the Liouville equation

$$z_{\tau\zeta} = e^z \quad (\text{A.15})$$

by differential substitutions

$$z = \phi + \ln(\phi_\zeta + \sqrt{\phi_\zeta^2 \pm 1}), \quad (\text{A.16})$$

which can be written in terms of hyperbolic functions as follows,

$$z = \phi + \text{Arsinh}\phi_\zeta \quad \text{or} \quad z = \phi + \text{Arcosh}\phi_\zeta. \quad (\text{A.17})$$

These substitutions map the general solutions (A.13,A.14) of the modified Liouville equations (A.10) into the general solution

$$z = \ln \frac{2F'(\tau)Q'(\zeta)}{(F(\tau) + Q(\zeta))^2}, \quad (\text{A.18})$$

of the Liouville equation (A.15), where

$$Q(\zeta) = -2/G(\zeta) - H(\zeta). \quad (\text{A.19})$$

Substitutions (A.17) can be reduced to a second order linear equation for the variable y defined by $y_\zeta/y = \frac{1}{2} \exp(z - \phi)$ which is considered as a function only of ζ :

$$y_{\zeta\zeta} - \left(\frac{Q''}{Q'} - \frac{2Q'}{F + Q} \right) y_\zeta \mp \frac{1}{4}y = 0. \quad (\text{A.20})$$

The above consideration gives one solution of this second order equation in the form

$$y = \exp \left(\frac{1}{2} \int \exp(z - \phi) d\zeta \right). \quad (\text{A.21})$$

At last we notice that Eq. (A.6) does not admit real substitution (A.17) (but only complex one) which transforms this equation into Liouville equation. Nevertheless, if we write the solution of the modified Liouville equation

$$\phi_{\tau\zeta} = -e^\phi \sqrt{1 - \varepsilon^2 \phi_\zeta^2} \quad (\text{A.22})$$

in the form

$$\phi = \ln \frac{2F'(\tau)G(\zeta)}{(F(\tau) - H(\zeta))^2 + \varepsilon^2 G^2(\zeta)}, \quad (\text{A.23})$$

where

$$H'^2 = G^2 - \varepsilon^2 G'^2,$$

then in the limit $\varepsilon \rightarrow 0$ it goes directly to the solution of the Liouville equation $\phi_{\tau\zeta} = -\exp \phi$.

References

- [1] S.A. Akhmanov, V.A. Vysloukh, and A.S. Chirkin, *Optics of Femtosecond Laser Pulses*, American Institute of Physics, New York, 1991.
- [2] D.L. Kaup, Stud. Appl. Math. **59**, 25 (1978)
- [3] F.G. Bass and V.G. Sinitsyn, Ukr. Fiz. J. **17**, 124 (1972)
- [4] S.A. Akhmanov, A.S. Chirkin, K.N. Drabovich, A.I. Kovrigin, R.V. Khokhlov, and A.P. Sukhorukov, IEEE J. Quantum Electronics, **QE-4**, 598 (1968)
- [5] H. Steudel, C. Figueira de Morrison Faria, M.G.A. Paris, A.M. Kamchatnov, and O. Steuernagel, Optics Commun. **150**, 363 (1998)
- [6] K.R. Khusnutdinova and H. Steudel, J. Math. Phys, **39**, 3754 (1998)
- [7] E. Hille, *Ordinary Differential Equations in the Complex Plane*, Wiley, New York, 1976.
- [8] E. Vessiot, J. Math. Pures et Appl., **21** 1 (1942)
- [9] A.V. Zhiber and V.V. Sokolov, Teor. Mat. Fiz., **120**, 20 (1999)
- [10] H. Steudel and D.J. Kaup, J. Modern Optics, **43**, 1851 (1996)
- [11] A.M. Kamchatnov and A.L. Chernyakov, Sov. J. Quantum Electr., **9**, 947 (1982)